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DIMENSIONS OF DIVISION RINGS

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ABSTRACT

Let k be a field. Write $D(G)$ for the quotient division ring of the group ring kG of a torsion-free, polycyclic-by-finite group G, and $D(\mathfrak{g})$ for the quotient ring of the enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$ over k. In this note we show that the Hirsch number $h(G)$ and $\dim_k \mathfrak{g}$ are invariants for the respective division rings, by calculating the Krull and global dimensions of $D(G) \otimes_k D(G)$ and $D(\mathfrak{g}) \otimes_k D(\mathfrak{g})$.

Let D be a division ring with a central subfield k. Recently, Resco has related the transcendence degree of D over k, that is, the maximal transcendence degree of (commutative) subfields of D, to the dimension of $D \otimes_k k(x_1, \cdots, x_n)$ (see [9] for the precise results). Here the dimension can be taken to be either Krull dimension in the sense of Rentschler–Gabriel or global dimension. This suggests that it may be profitable to study other overrings of D in order to understand the internal structure of D itself. We make one such approach in this note by determining, for various division rings D, the dimension of $D^{\mathrm{op}} \otimes D$.

More precisely, let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k, with enveloping algebra $U(\mathfrak{g})$, and G a torsion-free, polycyclic-by-finite group. Write $D(\mathfrak{g})$ and $D(G)$ for the quotient division rings of $U(\mathfrak{g})$ and kG, respectively. The main result of this note is the following:

THEOREM. (i) $\mathrm{Kdim}\, D(G)^{\mathrm{op}} \otimes_k D(G) = \mathrm{gldim}\, D(G)^{\mathrm{op}} \otimes_k D(G) = h(G)$, *the Hirsch number of* G;

(ii) $\mathrm{gldim}\, D(\mathfrak{g})^{\mathrm{op}} \otimes D(\mathfrak{g}) = \dim_k \mathfrak{g}$. *Furthermore*, $\mathrm{Kdim}\, D(\mathfrak{g})^{\mathrm{op}} \otimes D(\mathfrak{g}) = \dim \mathfrak{g}$, *provided that either* (a) $\mathfrak{g}$ *is solvable or* (b) k *is an uncountable field of characteristic zero and* $\mathfrak{g}$ *is algebraic.*

We conjecture that $\mathrm{Kdim}\, D(\mathfrak{g})^{\mathrm{op}} \otimes D(\mathfrak{g}) = \dim \mathfrak{g}$ will hold in all cases. The problem in the general case really stems from the fact that $\mathrm{Kdim}\, U(\mathfrak{g}) < \dim \mathfrak{g}$ will frequently hold.

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Note that the rings $D(G)$ and $D(\mathfrak{g})$ can be thought of as division rings of rational functions in $h(G)$ (or $\dim \mathfrak{g}$) non-commuting variables over k. The above theorem suggests, at least when $D^{\mathrm{op}} \otimes D$ is Noetherian, that $\dim D^{\mathrm{op}} \otimes D$ may give a better indication of the "size" of D than Resco's transcendence degree.

I would like to thank A. H. Schofield for stimulating my interest in this problem and R. L. Snider for several useful conversations. After this paper was submitted, Rosenberg's paper [10] appeared and has some overlap with the present paper. In particular, Rosenberg proves that $\operatorname{gldim} D \otimes D = 2n$, when D is the division ring of the n-th Weyl algebra.

1. On the dimension of $D^{\mathrm{op}} \otimes D$

We begin by giving various notation that will be used throughout this note. k will always be a field, $\mathfrak{g}$ a finite-dimensional Lie algebra over k and G a polycyclic-by-finite group. Note that kG and the enveloping algebra $U(\mathfrak{g})$ are Noetherian domains and so, by Goldie's Theorem, have quotient division rings. These will be denoted by $D(G)$ and $D(\mathfrak{g})$, respectively. Given a ring S, the global dimension of S will be written $\operatorname{gldim} S$ and the Krull dimension of S in the sense of Rentschler and Gabriel will be written $\operatorname{Kdim} S$. Since we will frequently not need to distinguish between them, $\operatorname{Dim} S$ will stand for either $\operatorname{gldim} S$ or $\operatorname{Kdim} S$. Similarly, D will denote either $D(G)$ or $D(\mathfrak{g})$ and R will denote either kG or $U(\mathfrak{g})$. Finally, $\otimes$ will always mean the tensor product over k.

The purpose of this section is to determine $\operatorname{Dim} D^{\mathrm{op}} \otimes D$. Note that, for each ring S that we consider, $S^{\mathrm{op}} \cong S$ and so we could equally well write $D \otimes D$ for $D^{\mathrm{op}} \otimes D$. We work with the latter as it seems to be slightly more natural.

We begin by producing an easy upper bound for $\operatorname{Dim} D^{\mathrm{op}} \otimes D$.

LEMMA 1.1. *Let $n = h(G)$, the Hirsch number of G, when $R = kG$ and let $n = \dim_k \mathfrak{g}$, when $R = U(\mathfrak{g})$. Then* $\operatorname{Dim} D^{\mathrm{op}} \otimes D \leqq \operatorname{Dim} D^{\mathrm{op}} \otimes R \leqq n$.

PROOF. The initial inequality of the lemma follows from the fact that $D^{\mathrm{op}} \otimes D$ is isomorphic to the localisation of $D^{\mathrm{op}} \otimes R$ at the set $\mathscr{C} = \{1 \otimes d : d \neq 0 \in R\}$. In order to prove the second inequality, we need to consider the various cases separately.

If $R = U(\mathfrak{g})$, then $D^{\mathrm{op}} \otimes R$ has an associated graded ring isomorphic to the polynomial ring $D[x_1, \cdots, x_n]$. Thus $\operatorname{Kdim} D^{\mathrm{op}} \otimes R \leqq n$ follows from [7]. By [4, proposition 2(iii)] and [1, theorem 8.2, p. 283], and in the notation of the latter,

$$\operatorname{gldim} D^{\mathrm{op}} \otimes R \leqq \operatorname{gldim} D^{\mathrm{op}} + \dim R = n.$$

Suppose next that $R = kG$. Then $D^{\mathrm{op}} \otimes R \cong DG$, the group ring. Thus $\operatorname{Kdim} D^{\mathrm{op}} \otimes R \leqq n$ follows from [12, theorem 2]. Secondly,

$$\operatorname{gldim} D^{\mathrm{op}} \otimes R \leqq \operatorname{gldim} D^{\mathrm{op}} + \dim R = \operatorname{gldim} kG = h(G),$$

where the three inequalities come from [4, proposition 2(iii)], [1, theorem 6.2, p. 195] and [5, theorem 3.6 and lemma 1.4], respectively.

The inverse inequalities to those given by Lemma 1.1 require somewhat more work and for these we will use the following "diagonal" embeddings of R into $D^{\mathrm{op}} \otimes D$.

NOTATION 1.2. Let $\hat{\mathfrak{g}} = \{1 \otimes g - g \otimes 1 \in U(\mathfrak{g})^{\mathrm{op}} \otimes U(\mathfrak{g}) : g \in \mathfrak{g}\}$ and $\hat{R} = \hat{U}$, the subalgebra of $R^{\mathrm{op}} \otimes R$ generated by $\hat{\mathfrak{g}}$. Similarly, set

$$\hat{G} = \{g^{-1} \otimes g \in kG^{\mathrm{op}} \otimes kG : g \in G\}$$

and write $\hat{R} = \widehat{kG}$ for the subalgebra of $kG^{\mathrm{op}} \otimes kG$ generated by $\hat{G}$. Finally write $\hat{\mathfrak{a}}$ for the augmentation ideal of $\hat{R}$; so $\hat{\mathfrak{a}}$ is generated by the $\{g \in \hat{\mathfrak{g}}\}$ if $\hat{R} = \hat{U}$ and by the $\{1 - g : g \in \hat{G}\}$ if $\hat{R} = \widehat{kG}$.

The main theorem will be proved by bounding $\operatorname{Dim} D^{\mathrm{op}} \otimes D$ from below by $\operatorname{Dim} \hat{R}$. For this we need some more detailed information about the structure of $D^{\mathrm{op}} \otimes D$ as an $\hat{R}$-module.

LEMMA 1.3. (a) $\hat{R} \cong R$.

(b) *Let* $R' = \{a \otimes 1 : a \in R^{\mathrm{op}}\} \subset R^{\mathrm{op}} \otimes R$ *and* $D' = \{d \otimes 1 : d \in D^{\mathrm{op}}\} \subset D^{\mathrm{op}} \otimes R$. *Then, as k-algebras,* $R^{\mathrm{op}} \otimes R = R' . \hat{R}$ *and* $D^{\mathrm{op}} \otimes R = D' . \hat{R}$.

(c) *Let* $\{d_i : i \in I\}$ *be a k-basis for* D^{op}. *Then* $D^{\mathrm{op}} \otimes R$ *is a free (left or right)* $\hat{R}$*-module, with basis* $\{d_i \otimes 1 : i \in I\}$.

PROOF. (a) We will give the proof of this part of the lemma in some detail, as the same argument can then be used in part (c). We begin with the group ring case. Certainly $\hat{G} \cong G$ and so by the universality of the group ring, this induces a ring homomorphism from kG onto $\widehat{kG}$. If $r = \Sigma\{k_i g_i : k_i \in k$ and $g_i \in G\}$ is a non-zero element of kG, then its image $\hat{r} = \Sigma\, k_i (g_i^{-1} \otimes g_i)$ in $\widehat{kG}$ is clearly non-zero. Thus $\widehat{kG} \cong kG$.

Now turn to the case of Lie algebras. Again, from the universality of the enveloping algebra, there exists a homomorphism from $U(\mathfrak{g})$ onto $\hat{U}$. However, it is not now advisable to explicitly calculate elements of $\hat{U}$. Instead, regard elements of $R^{\mathrm{op}} \otimes R$ as polynomials in the $\{1 \otimes g : g \in \mathfrak{g}\}$ with coefficients in R'. Now, given a monomial v in $U(\mathfrak{g})$, then its image $\hat{v}$ in $\hat{U}$ will be some polynomial whose leading term is $1 \otimes v$. So, if $r \neq 0 \in U(\mathfrak{g})$, then its image $\hat{r}$ in $\hat{U}$ will have a non-zero leading term and so will be non-zero.

(b) Clearly $R'.\hat{R}$ contains $1\otimes g$ for all $g\in G$ (respectively $g\in \mathfrak{g}$) and the result follows.

(c) By symmetry, it suffices to show that the $\{d_i\otimes 1\}$ form a basis for $D^{\mathrm{op}}\otimes R$ as a right $\hat{R}$-module. Certainly they span the module. Thus it remains to show that $\Sigma(d_i\otimes 1)r_i=0$ for some $r_i\in\hat{R}$ implies that each $r_i=0$. If $R=kG$, then this is a straightforward calculation. If $R=U(\mathfrak{g})$, then it follows from the same leading term argument that was used in part (a).

Corollary 1.4. *$D^{\mathrm{op}}\otimes R$ is faithfully flat as a (left or right) $\hat{R}$-module. Thus $D^{\mathrm{op}}\otimes D$, being a localisation of $D^{\mathrm{op}}\otimes R$, is flat as both a left and a right $\hat{R}$-module.*

It would be interesting to know whether $D^{\mathrm{op}}\otimes D$ is faithfully flat as an $\hat{R}$-module. However, it is sufficient for our purposes to know that $\hat{\mathfrak{a}}(D^{\mathrm{op}}\otimes D)\neq D^{\mathrm{op}}\otimes D$, which is easy to ascertain. Observe that D is naturally a right $D^{\mathrm{op}}\otimes D$-module under the action $d(a\otimes b)=adb$, for $d\in D$ and $a\otimes b\in D^{\mathrm{op}}\otimes D$. Write $\operatorname{aug}(D^{\mathrm{op}}\otimes D)$ for the right annihilator in $D^{\mathrm{op}}\otimes D$ of the element $1\in D$. Equivalently, $\operatorname{aug}(D^{\mathrm{op}}\otimes D)=\{\Sigma f_i\otimes g_i\in D^{\mathrm{op}}\otimes D:\Sigma f_ig_i=0\}$. The next lemma is now immediate.

Lemma 1.5. *With the above notation, $\hat{\mathfrak{a}}(D^{\mathrm{op}}\otimes D)\subseteq\operatorname{aug}(D^{\mathrm{op}}\otimes D)\neq D^{\mathrm{op}}\otimes D$.*

This result illustrates why we prefer to work with $D^{\mathrm{op}}\otimes D$ rather than $D\otimes D$. For, given an arbitrary division ring E, it is not clear whether E can be made into a right $E\otimes E$-module. The following strengthening of Lemma 1.5 will not be needed subsequently, but may be of some independent interest.

Corollary 1.6. $\hat{\mathfrak{a}}(D^{\mathrm{op}}\otimes D)=\operatorname{aug}(D^{\mathrm{op}}\otimes D)$.

Proof. We first show that, if $a,b\in R$, then $a\otimes b\equiv ab\otimes 1 \bmod \hat{\mathfrak{a}}(R^{\mathrm{op}}\otimes R)$. So, suppose that $R=U(\mathfrak{g})$ and that b is a monomial; say, $b=g_1g_2\cdots g_m$ for some $g_i\in\mathfrak{g}$. As each $1\otimes g_i-g_i\otimes 1\in\hat{\mathfrak{a}}$, we have

$$\begin{aligned} a\otimes b &\equiv a\otimes b+(g_1\otimes 1-1\otimes g_1)(a\otimes g_2\cdots g_m) && \bmod \hat{\mathfrak{a}}(R^{\mathrm{op}}\otimes R)\\ &\equiv ag_1\otimes g_2\cdots g_m\\ &\equiv\cdots\equiv ag_1g_2\cdots g_m\otimes 1 && \bmod \hat{\mathfrak{a}}(R^{\mathrm{op}}\otimes R).\end{aligned}$$

Thus, for any $b\in U(\mathfrak{g})$, one obtains $a\otimes b\equiv ab\otimes 1 \bmod \hat{\mathfrak{a}}(R^{\mathrm{op}}\otimes R)$. A similar proof works when $R=kG$.

Now suppose that $h\in\operatorname{aug}(D^{\mathrm{op}}\otimes D)$. By multiplying h by an appropriate element of $\mathscr{C}=\{a\otimes b: a,b\neq 0\in D\}$, we may assume that $h\in R^{\mathrm{op}}\otimes R$. By replacing h by $h - x$ for some $x \in \hat{\mathfrak{a}}(R^{\mathrm{op}} \otimes R)$ and using the result of the last paragraph, we may suppose that $h = h' \otimes 1$ for some $h' \in R^{\mathrm{op}}$. Clearly $h' \otimes 1 \in \mathrm{aug}(D^{\mathrm{op}} \otimes D)$ if and only if $h' = 0$. This implies that $h \in \hat{\mathfrak{a}}(D^{\mathrm{op}} \otimes D)$, as required.

The computation of Dim $D^{\mathrm{op}} \otimes D$ now follows by standard methods from the literature, although the different cases will require slightly different techniques. Write $\mathrm{pd}_S M$ and w-$\mathrm{pd}_S M$ for the projective dimension and weak homological dimension of an S-module M.

Proposition 1.7. $\mathrm{Gldim}\, D(G)^{\mathrm{op}} \otimes D(G) = h(G)$, *the Hirsch number of* G *and* $\mathrm{gldim}\, D(\mathfrak{g})^{\mathrm{op}} \otimes D(\mathfrak{g}) = \dim_k \mathfrak{g}$.

Proof. Let $n = h(G)$, respectively $n = \dim_k \mathfrak{g}$. By Lemma 1.1, $\mathrm{gldim}\, D^{\mathrm{op}} \otimes D \leqq n$. Set $N = \hat{R}/\hat{\mathfrak{a}}$. Then $\mathrm{pd}_{\hat{R}} N = \mathrm{gldim}\, \hat{R} = n$, by [1, theorem 8.2, p. 283] if $R = U(\mathfrak{g})$ and by [5, theorem 3.6] if $R = kG$. Since $D^{\mathrm{op}} \otimes D$ is a flat $\hat{R}$-module (Corollary 1.4),

$$M = N \otimes_{\hat{R}} (D^{\mathrm{op}} \otimes D) \cong D^{\mathrm{op}} \otimes D / \hat{\mathfrak{a}}(D^{\mathrm{op}} \otimes D).$$

Further, $M \neq 0$, by Lemma 1.5. Thus the natural homomorphism $N \to M$ given by $n \to n \otimes 1$ is non-zero which, as N is simple, implies that it is injective. So there exists a short exact sequence of $\hat{R}$-modules

$$0 \to N \to M \to T \to 0.$$

Since w-pd $N = n = \mathrm{gldim}\, \hat{R} \geqq$ w-pd T ([11, theorem 9.22, p. 241]), this implies that w-$\mathrm{pd}_{\hat{R}} M = n$. So, by [11, ex. 3.38, p. 94 and theorem 9.13, p. 239],

$$\text{w-pd}_{D^{\mathrm{op}} \otimes D} M \geqq \text{w-pd}_{\hat{R}} M \geqq n,$$

as required.

In order to obtain information about the Krull dimension of $D^{\mathrm{op}} \otimes D$, we will use Resco's notion of an r-sequence. A set of elements $\{a_1, \cdots, a_n\}$ of a ring A will form an *r-sequence* provided that (i) $\Sigma a_i A \neq A$, (ii) $a_u (\Sigma_1^{u-1} a_i A) \subseteq \Sigma_1^{u-1} a_i A$ for each $u > 1$, (iii) a_1 is right regular and (iv) if $u > 1$ and $a_u f \in \Sigma_1^{u-1} a_i A$, then $f \in \Sigma_1^{u-1} a_i A$. The basic properties of r-sequences may be found in [8].

Proposition 1.8. $\mathrm{Kdim}\, D(G)^{\mathrm{op}} \otimes D(G) = h(G)$.

Proof. Once again, by Lemma 1.1 it suffices to show that $\mathrm{Kdim}\, D^{\mathrm{op}} \otimes D \geqq n = h(G)$. There exists a subnormal chain of subgroups

$$H_0 = (1) \lhd H_1 \lhd \cdots \lhd H_n = H \lhd \hat{G},$$

such that each $H_i/H_{i-1} = \langle h_i + H_{i-1}\rangle$ is infinite cyclic and $\hat{G}/H$ is finite. Given $u > 1$, note that $kH_u/\Sigma_1^{u-1}(1-h_i)kH_u \cong k(H_u/H_{u-1})$ is a domain. It follows easily from this observation that $\{1-h_i : 1 \leqq i \leqq n\}$ is an r-sequence in kH. Since $D^{op}\otimes R$ is a free $\widehat{kG}$-module (Lemma 1.3) and $\widehat{kG}$ is a free kH-module, it follows from [8, proposition 1.8(i)] that $\{1-h_i\}$ forms an r-sequence in $D^{op}\otimes R$. But $D^{op}\otimes D$ is a localisation of $D^{op}\otimes R$ and, by Lemma 1.5, $\Sigma(1-h_i)D^{op}\otimes D \neq D^{op}\otimes D$. So, by [8, proposition 1.3], $\{1-h_i\}$ forms an r-sequence in $D^{op}\otimes D$. Finally, by [8, proposition 1.4], this implies that $\operatorname{Kdim} D^{op}\otimes D \geqq n$, as required.

PROPOSITION 1.9. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k, and suppose that either*

(i) *$\mathfrak{g}$ is solvable, or*

(ii) *k is an uncountable field of characteristic zero and $\mathfrak{g}$ is algebraic.*

Then $\operatorname{Kdim} D(\mathfrak{g})^{op}\otimes D(\mathfrak{g}) = \dim_k \mathfrak{g}$.

PROOF. As usual, we need only show that $\operatorname{Kdim} D^{op}\otimes D \geqq n = \dim \mathfrak{g}$. Suppose that $\mathfrak{g}$ is a solvable Lie algebra. Then there exists a chain of subalgebras

$$0 = \mathfrak{h}_0 \subset \mathfrak{h}_1 = kx_1 \subset \cdots \subset \mathfrak{h}_n = \mathfrak{h}_{n-1} + kx_n = \mathfrak{g},$$

such that, for $0 \leqq i \leqq n-1$, $\mathfrak{h}_i$ is a Lie ideal of $\mathfrak{h}_{i+1}$ of codimension one. It is routine to show that $\{x_1, \cdots, x_n\}$ form an r-sequence of length n in $U(\mathfrak{g})$. As in the proof of Proposition 1.8, this quickly leads to $\operatorname{Kdim} D^{op}\otimes D = n$.

Let U_2 be the two-dimensional, non-abelian, solvable Lie algebra and I the one-dimensional Lie algebra. Let $\mathfrak{h}$ be the Lie algebra formed as the direct sum of u copies of U_2 and v copies of I. Then, by the result of the last paragraph, $\operatorname{Kdim} D(\mathfrak{h})^{op}\otimes D(\mathfrak{h}) = \dim \mathfrak{h} = 2u + v$. But $D(\mathfrak{h})$ is isomorphic to the quotient division ring of the polynomial extension $A_{u,v} = A_u[z_1, \cdots, z_v]$ of the u-th Weyl algebra A_u. In particular, $\operatorname{Kdim} D(\mathfrak{g})^{op}\otimes D(\mathfrak{g}) = \dim \mathfrak{g}$ will hold for any Lie algebra $\mathfrak{g}$ for which the Gel'fand–Kirillov conjecture holds.

If $\mathfrak{g}$ is an algebraic Lie algebra over an uncountable field k of characteristic zero, then [6, theorem 4.1(ii)] is close enough to a solution of this conjecture to complete our proof. For, let K be the centre of $D(\mathfrak{g})$, with algebraic closure F, and set $u = \frac{1}{2}(\dim \mathfrak{g}$ — transcendence degree$(K))$, which is an integer. Given an Ore domain A, write $D(A)$ for the quotient division ring of A. Then [6] shows that $D(D(\mathfrak{g})\otimes_K F)$ embeds in $D(A_u(F))$ in such a way that $D(A_u(F))$ is a finite-dimensional vector space over $D(D(\mathfrak{g})\otimes_K F)$. Since $D(\mathfrak{g})$ is a finitely generated division algebra over K, it follows that there exists a finite field extension L of K such that $D = D(\mathfrak{g})$ embeds in $D_1 = D(A_u(L))$, again with finite index. In particular, $D_1^{op} \otimes D_1$ is a finitely generated $D^{op} \otimes D$-module. So, using the comments of the last paragraph,

$$\mathrm{Kdim}\, D^{op} \otimes D \geqq \mathrm{Kdim}_{D^{op}\otimes D} (D_1^{op} \otimes D_1) \geqq \mathrm{Kdim}\, D_1^{op} \otimes D_1 = \dim \mathfrak{g},$$

as required.

The Theorem of the introduction is just the combination of the last three propositions.

We conjecture that Proposition 1.9 will hold for any finite-dimensional Lie algebra $\mathfrak{g}$. Ironically, the problem really stems from a case we have already solved. For, let $\mathfrak{g}$ be a (non-abelian) semi-simple Lie algebra over a field k of characteristic zero. Then, by [13], $\mathrm{Kdim}\, U(\mathfrak{g}) < \dim_k \mathfrak{g}$. So, it is not clear how one should in general construct one's r-sequences. Observe, however, that if $D = D(\mathfrak{g})$, then one can construct the Lie algebra $\mathfrak{g} \otimes D$ over D. Since $U(\mathfrak{g} \otimes D) \cong U(\mathfrak{g}) \otimes D$, it follows from Proposition 1.9 and Lemma 1.1 that

$$\mathrm{Kdim}\, U(\mathfrak{g} \otimes D) = \mathrm{Kdim}\, U(\mathfrak{g}) \otimes D = \dim_D \mathfrak{g} \otimes D.$$

In other words, the problems of calculating $\mathrm{Kdim}\, U(\mathfrak{g})$ for a semi-simple Lie algebra $\mathfrak{g}$ disappear if one defines the Lie algebra over an appropriate division ring.

One can ask what is the dimension of the tensor product $\bigotimes_1^t D$ of $t \geqq 3$ copies of $D = D(G)$ or of $D = D(\mathfrak{g})$. This problem seems to be more difficult and to depend on more than one invariant of the group ring or enveloping algebra. For the quotient division ring $D_{u,v}$ of the generalised Weyl algebra $A_{u,v}$ (and therefore for $D(\mathfrak{g})$ when $\mathfrak{g}$ is algebraic) it is possible to determine the answer and we announce:

Proposition 1.10. *Let k be a field of characteristic zero. Then, for any $t \geqq 3$ and any integers u and v,*

$$\mathrm{Kdim} \bigotimes_1^t D_{u,v} = \mathrm{gldim} \bigotimes_1^t D_{u,v} = tu + (t-1)v.$$

We will not prove this result. However, we will observe that, as $\otimes D_{u,v}$ is a localisation of $A_{tu,(t-1)v}(D_{0,v})$, certainly $\dim \otimes D_{u,v} \leqq tu + (t-1)v$. The other inequality is obtained by constructing a suitable r-sequence.

We end with an amusing example.

Example 1.11. *There exist division rings D and E such that, for either Krull or global dimension,*

$$\mathrm{Dim}\, D \otimes_k E = 1 < \mathrm{Dim}\, D \otimes D = 2 < \mathrm{Dim}\, E \otimes E = 3.$$

Further, each ring is Noetherian.

PROOF. Take $D = D_{1,0}$, the quotient division ring of the first Weyl algebra over a field k of characteristic zero and E the field of rational functions in 3 variables over k. The result is now obvious.

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